Convex hypersurfaces in Hadamard manifolds.

A.A.Borisenko.

Abstract. We prove the theorem about extremal property of Lobachevsky space among simply connected Riemannian manifolds of nonpositive curvature.

Hadamard proved the following theorem. Let φ be an immersion of a compact oriented n-dimensional manifold M in Euclidean space $E^{n+1}\,(n \geqslant\ 2)$ with everywhere positive Gaussian curvature. Then $\varphi(M)$ is a convex hypersurface [1].

Chern and Lashof [2] generalized this theorem. Let φ be an immersion of a compact oriented n-dimensional manifold M in E^{n+1}. Then the following two assertions are equivalent:

(i) The degree of the spherical mapping equals ± 1, and the Gaussian curvature does not change sign (i.e., it is everywhere nonnegative or everywhere nonpositive);

(ii) $\varphi(M)$ is a convex hypersurface.

By Gaussian curvature, we mean the product of the principal curvatures.

S.Alexander generalized Hadamard theorem for compact hypersurfaces in any complete, simply connected Riemannian manifold of nonpositive sectional curvature.[3]

A topological immersion $f : N^n \to M$ of a manifold N^n into a Riemannian manifold M is called *locally convex* at a point $x \in N^n$ if has a neighbourhood U such that $f(U)$ is a part of the boundary of a convex set in M.

Heijenoort proved the following theorem. Let $f : N^n \to E^{n+1}$, where $n \geq 2$, be a topological immersion of a connected manifold N^n. If f is locally convex at all points and has at least one point of local strict support and N^n is complete in the metric induced by immersion, then f is an embedding and $F = f(N^n)$ is the boundary of a convex body [4].

In [5] this theorem was generalized to h-locally convex (i.e., such that each point has a neighbourhood lying on one side from a horosphere) regular hypersurfaces in Lobachevsky space and in [6], to nonregular hypersurfaces.

In this section we shall recall some definitions and we shall state the notation.

A *Hadamard manifold* is a complete simply connected Riemannian manifold with sectional curvature $K \leq 0$.

Like in the hyperbolic space, a *horoball in a Hadamard manifold* M is the domain obtained as the limit of the balls with their centres in a geodesic ray going to infinity, and their corresponding geodesic spheres containing a fixed point. The boundary of a horoball is a *horosphere*. In general, a horosphere is a C^2 hypersurface. An *h-convex set* in a Hadamard manifold M of dimension $n+1$ is a subset $\Omega \subset M$ with boundary $\partial\Omega$ satisfying that, for every $P \in \partial\Omega$ there is a horosphere H of M through P such that Ω is locally contained in the horoball of M bounded by H. This H is called a *supporting horosphere* of Ω (and $\partial\Omega$).

For Hadamard manifolds M satisfying $-k_1^2 \geqslant K \geqslant -k_2^2$, $k_1, k_2 > 0$, if H is horosphere, at each point of H where the normal curvature k_n is well defined, it satisfies $k_1 \leq k_n \leq k_2$.

For geodesic spheres of radius r normal curvatures satisfy inequality

$$k_1 \coth k_1 r \leq k_n \leq k_2 \coth k_2 r.$$

Note that the value $k \coth kr$ is the geodesic curvature of a circumference of radius r in Lobachevsky plane of curvature $-k^2$.

An orientable regular (C^2 or more) hypersurface F of a Hadamard manifold M is λ-convex if, for a selection of its unit normal vector, the normal curvature k_n of F satisfies $k_n \geqslant \lambda$. A domain $\Omega \subset M$ is λ-convex if for every point $P \in \partial\Omega$ there is a regular λ-convex hypersurface F through P leaving a neighbourhood of P in the convex side (the side where the unit normal vectors points) of F. If $\partial\Omega$ is regular, then it is a regular λ-convex hypersurface.

Given any set $\Omega \subset M$, an *inscribed ball (inball* for short) is a ball in M contained in Ω with maximum radius. Its radius is called the *inradius* of Ω, and it will be always denoted by r. Moreover, we shall denote by O the (not necessarily unique) centre of an inball of Ω, and by d the distance, in M to O.

A circumscribed ball (or circumball) is a ball in M^{n+1} containing Ω with minimum radius. Its radius is called circumradius in Ω and notes by R.

Now we shall prove the following theorems.

Theorem. 1 *Let M^{n+1} be a simply connected complete Riemannian manifold with sectional curvature*

$$-k_1^2 \geqslant K \geqslant -k_2^2, \quad k_2 \geqslant k_1 > 0.$$

Suppose that $F \subset M^{n+1}$ be a complete immersed hypersurface with normal curvatures

$$k_n \geqslant k_2.$$

Then either

I) F^n is a compact convex hypersurface diffeomorphic to the sphere S^n and

$$R - r < k_2 \ln 2$$

or

II) F^n is a horosphere in M^{n+1} and ambient space M^{n+1} is a hyperbolic space of constant curvature $-k_2^2$.

For more strong condition on the normal curvatures F^n it is true

Theorem. 2 *Let M^{n+1} be a Hadamard manifold with sectional curvature*

$$-k_1^2 \geqslant K \geqslant -k_2^2, \quad k_2 \geqslant k_1 \geqslant 0.$$

Let F^n be a complete immersed hypersurface with normal curvatures more or equal $k_2 \coth k_2 r_0$ at any point F^n. Then either

I) F^n is a compact convex hypersurface diffeomorphic sphere S^n and radius of circumscribed ball

$$R < r_0$$

or

II) *F^n is a sphere of radius r_0 which is boundary the ball Ω. The ball Ω is isometric to the ball of radius r_0 of hyperbolic space with constant curvature $-k_2^2$*

The ambient space M^{n+1} is a C^3 regular Riemannian manifold. For proof of part II) of theorem (1) we need the condition.

$$|\nabla R| \leq C,$$

where $|\nabla R|$ is a covariant differential of curvature tensor M^{n+1}, C is a positive constant.

For condition $K_\sigma \leq -k_1^2 < 0$ and $|\nabla R| < C$ a horosphere is C^3 regular hypersurface and manifold M^{n+1} is C^2-regular Riemannian manifold in horospheric coordinates [7].

At any point smooth hypersurface F^n in Hadamard manifold there are two tangent horospheres. Let normal curvatures F^n at some point $P \in F^n$ with respect some normal be greater zero, one of the horospheres with positive normal curvature with respect the same normal we call *tangent horosphere.*

Proof of the theorem 1.

I). From the condition of the theorem it follows that normal curvature of the horosphere H^n in M^{n+1}

$$k_n/_{H^n} \leq k_2.$$

And for every point $P \in F^n$, normal curvatures of tangent horosphere in

the corresponding directions satisfy the inequality

$$k_n(a)/_{F^n} \geqslant k_n(a)/_{H^n}.$$

Suppose that in point P_0 it is true the strong inequality

$$k_n(a)/_{F^n} > k_n(a). \tag{1}$$

Let n_0 be the unit normal at the point P_0, such that the normal curvatures of F^n at the point $P_0 \in F^n$ with respect normal n_0 are positive, H^n be a tangent horosphere at the point P_0 with the normal n_0. From the inequality (1) it follows that there exists some neighbourhood of the point P_0 on F^n such that it lies inside the horoball bounded by horosphere H^n. Let we take a horospherical system of the coordinates in M^{n+1} with the base H^n, t is a length parameter along geodesic line orthogonal H^n, positive direction coincides with the normal n_0 at the point P_0. From another side t is a distance from a point of M^{n+1} to the horosphere H^n. Let the function $f = t$ be the restriction t oh the hypersurface F^n, at the point P_0 the function f has a strong minimum. Let φ be the angle between the direction $\frac{\partial}{\partial t}$ and the unit normal N of the hypersurface F^n. Along integral curves of the vector field $X = grad\, f/_{F^n}$ on the hypersurface F^n the angle φ satisfies the equation [8].

$$k_n = \mu \cos\varphi + \sin\varphi \frac{d\varphi}{dt}, \tag{2}$$

where k_n is the normal curvature F^n in the direction X at the point $P \in F^n$, μ is the normal curvature of the coordinate horosphere at the point $P \in F^n$ in the direction Y, which is orthogonal projection the vector X on the tangent space of the coordinate horosphere at the point P.

As $k_n \geqslant k_2$ and normal curvatures of the horosphere $\mu \leq k_2$ that from (2)

for $\varphi \leq \frac{\pi}{2}$ it follows

$$k_2(1-\cos\varphi) \leq \sin\varphi \tfrac{d\varphi}{dt};$$

$$\frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}} \geqslant e^{\frac{k_2}{2}(t-t_0)};$$

$$\sin\tfrac{\varphi}{2} \geqslant \sin\tfrac{\varphi_0}{2} e^{\frac{k_2}{2}(t-t_0)},$$

where $\varphi_0 > 0$ is the angle between $\frac{\partial}{\partial t}$ and the normal N for small t_0. It follows from inequality (1) at the point P_0. The angle φ monotonically increases along integral curve and for

$$t \leq \frac{2}{k_2} \ln \frac{e^{\frac{k_2 t_0}{2}}}{\sqrt{2}(\sin\frac{\varphi_0}{2})}$$

reaches the value $\frac{\pi}{2}$. For $\varphi \geqslant \frac{\pi}{2}$ we have

$$k_2 \leq \sin\varphi \frac{d\varphi}{dt};$$

$$\cos\varphi \leq 1 - k_2(t - t_1),$$

where $\varphi(t_1) = \frac{\pi}{2}$ and for $t_2 \leq t_1 + \frac{2}{k_2}$ the angle φ reaches the value π and function $f = t/_{F^n}$ at this point achieves strong maximum.

The length of integral curve on the hypersurface F^n of the vector field $X = grad\, f/_{F^n}$ satisfies the inequality

$$k_2(1-\cos\varphi) \leq \frac{d\varphi}{ds};$$

$$s \leq s_0 + \frac{\cot\frac{\varphi_0}{2}}{k_2}.$$

It follows that point Q_0, where $\varphi = \pi$ does not go to infinity. Let t_2 be the infinum of the value t on integral curves of vector field $X = grad\, f/_{F^n}$ such that $\varphi(t_2) = \pi$. Level hypersurfaces of the function $f = t$ for $0 < t < t_2$

are spheres S^{n-1} and points P_0 and Q_0 are strong minimum and strong maximum and function f is a Morse function on F^n with two critical points. Therefore the hypersurface F^n is homeomorphic to sphere S^n. From the condition $k_n \geqslant k_2$ we obtain that second quadratic form F^n is positive definite at any point. From theorem S. Alexander [4] it follows that F^n is embedded compact convex hypersurface diffeomorphic S^n and bounds convex domain Ω. The domain Ω is k_2-convex and satisfies the condition of theorem 3.1 [9] and

$$\max d(O, \partial\Omega) < k_2 \ln 2,$$

where O is the centre of the inscribed ball.

II)1). Suppose that at any point $P \in F^n$ there exists the direction $a \in T_pF^n$ such that

$$k_n(a)/_{F^n} = k_n(a)/_{H^n}.$$

Let show that some neighbourhood $U \subset F^n$ of a point $P_0 \in F^n$ lies in the horoball bounded by tangent horosphere H^n. Let take horospherical system of coordinate with the base H^n.

The metric M^{n+1} has the form

$$ds^2 = dt^2 + g_{ij}(t,\theta)d\theta^i d\theta^j. \tag{3}$$

The equation of the hypersurface F^n in the neighbourhood $P_0 \in F^n$ is

$$t = \rho(\theta).$$

The unit normal vector N to F^n has coordinates

$$\xi^k = -\frac{\rho^k}{\sqrt{1 + \langle grad\,\rho, grad\,\rho\rangle}}, \quad k = 1, \dots, n \tag{4}$$

$$\xi^{n+1} = \frac{1}{\sqrt{1 + \langle grad\,\rho, grad\,\rho\rangle}},$$

where

$$\rho^k = g^{ks}\rho_s, \quad \langle grad\,\rho, grad\,\rho\rangle = g_{ij}\rho^i\rho^j, \tag{5}$$

$$\rho_i = \frac{\partial \rho}{\partial \theta^i}.$$

Coefficient of the second fundamental form of F^n is equal [10]

$$\Omega_{ij} = \cos\varphi \left[\rho_{i,j} - \frac{1}{2}\frac{\partial g_{ij}}{\partial t} - \frac{1}{2}\frac{\partial g_{jk}}{\partial t}\rho_i\rho^k - \frac{1}{2}\frac{\partial g_{ik}}{\partial t}\rho_j\rho^k\right], \qquad (6)$$

where φ is the angle between $\frac{\partial}{\partial t}$ and normal N,

$$\cos\varphi = \frac{1}{\sqrt{1 + \langle grad\,\rho, grad\,\rho\rangle}}, \quad \rho_{i,j} = \rho_{ij} - \Gamma^k_{ij/g}\rho_k, \qquad (7)$$

where $\Gamma^k_{ij/g}$ are Kristoffel symbols of the metric

$$d\sigma^2 = g_{ij} d\theta^i\, d\theta^j.$$

Coefficients of metric tensor F^n have the form

$$a_{ij} = g_{ij} + \rho_i\rho_j. \qquad (8)$$

From the conditions of the theorem normal curvatures

$$k_n/_{F^n} \geqslant k_2.$$

And from (6) it follows that for any tangent vector $b \in F^n$,
$b = (b^1, \dots, b^n)$.

$$\cos\varphi \left[\rho_{i,j} b^i b^j - \frac{1}{2}\frac{\partial g_{ij}}{\partial t} b^i b^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial t}\rho_i b^i \rho^k b^j - \frac{1}{2}\frac{\partial g_{ik}}{\partial t}\rho_j b^j \rho^k b^i\right] \geqslant$$
$$\geqslant k_2(g_{ij} b^i b^j + (\rho_i b^i)^2). \qquad (9)$$

Let introduce the function $h = e^{k_2\rho(\theta)}$.

$$h_i = k_2 e^{k_2\rho}\rho_i;$$

$$h_{ij} = k_2^2 e^{k_2\rho}\rho_i\rho_j + k_2 e^{k_2\rho}\rho_{ij}.$$

Hence

$$\rho_i = \frac{h_i}{h}\frac{1}{k_2};$$

$$\rho_{ij} = \frac{1}{k_2}\frac{h_{ij}}{h} - \frac{1}{k_2}\frac{h_i}{h}\frac{h_j}{h}; \tag{10}$$

$$\rho_{i,j} = \frac{1}{k_2}\frac{hh_{i,j} - h_i h_j}{h^2}.$$

And inequality (9) we rewrite in the following way:

$$\cos\varphi\left[\frac{1}{k_2}hh_{i,j}b^ib^j - \frac{1}{k_2}(h_ib^i)^2 - \frac{1}{2}\frac{\partial g_{jk}}{\partial t}b^ib^jh^2 - \frac{1}{2k_2^2}\frac{\partial g_{ik}}{\partial t}h_ib^ih^kb^j - \right.$$
$$\left.\frac{1}{2k_2^2}\frac{\partial g_{ik}}{\partial t}h_jb^jh^kb^i\right] \geqslant k_2\left[h^2g_{ij}b^ib^j + \frac{1}{k_2^2}(h_ib^i)^2\right] \tag{11}$$

Since the normal curvature of horosphere in M^{n+1} less or equal k_2 than

$$-\frac{1}{2}\frac{\partial g_{ij}}{\partial t}b^ib^j \leq k_2 g_{ij}b^ib^j, \tag{12}$$

where $A_{ij} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial t}$ are coefficients of the second fundamental form of horosphere $t = const.$

$$\left|-\frac{1}{2}\frac{\partial g_{jk}}{\partial t}h_ib^ih^kb^j\right| = (A_{jk}h^kb^j)|h_ib^i| \leq$$
$$\leq \sqrt{(A_{jk}h^kh^j)A_{jk}b^kb^j}|h_ib^i| \leq k_2|grad\,h||b||h_ib^i|, \tag{13}$$

where $|b|^2 = g_{ij}b^ib^j, \quad |grad\,h|^2 = g_{ij}h^ih^j.$
Let we substitute (12), (13) in (11) and obtain

$$\cos\varphi\frac{1}{k_2}hh_{i,j}b^ib^j \geqslant k_2h^2(1-\cos\varphi)|b|^2 + \frac{1}{k_2}(1+\cos\varphi)(h_ib^i)^2 -$$
$$-2\frac{1}{k_2}|grad\,h||b||(h_ib^i)|. \tag{14}$$

The expression in the right side in the quadratic equation with respect $|(h_i b^i)|$. The discriminant of this equation is

$$\frac{1}{k_2^2}|grad\, h|^2|b|^2 - h^2 \sin^2\varphi |b|^2. \tag{15}$$

But

$$\cos^2\varphi = \frac{1}{1+|grad\,\rho|^2} = \frac{k_2^2 h^2}{k_2^2 h^2 + |grad\, h|^2},$$

$$\sin^2\varphi = \frac{|grad\, h|^2}{k_2^2 h^2 + |grad\, h|^2}.$$

And we rewrite (15) in the form

$$\frac{|b|^2}{k_2^2}\left(\frac{|grad\, h|^4}{k_2^2 h^2 + |grad\, h|^2}\right) \geqslant 0. \tag{16}$$

From (14) it follows

$$h_{i,j} b^i b^j \geqslant 0. \tag{17}$$

Let L be lines on F^n which satisfy the system of the equations

$$\frac{\partial^2\theta^k}{\partial s^2} + \Gamma^k_{ij/g}(\theta, \rho(\theta))\frac{\partial\theta^i}{\partial s}\frac{\partial\theta^s}{\partial s} = 0. \tag{18}$$

From any point and in any direction goes through only one line from this family. These line we call g-geodesic. We take the restriction the function h on this line

$$\theta^i = \theta^i(s);$$

$$h_s = h_i \frac{d\theta^i}{ds};$$

$$h_{ss} = h_{ij}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds} + h_k \frac{d^2\theta^k}{d^2 s}. \tag{19}$$

If we substitute (18) in (19) then

$$h_{ss} = h_{i,j}\frac{d\theta^i}{ds}\frac{d\theta^j}{ds} \geqslant 0. \tag{20}$$

At the point $P_0 \quad h = 1, h_s = 0$ and from (20) it follows that along g-geodesic lines which go through the point P_0, $h \geqslant 1$. On tangent horosphere H^n, $h = 1$ and the hypersurface F^n lies from one side tangent horosphere H^n.

2). Let P_0 be an arbitrary fixed point F^n, $H^n(P_0)$ — tangent horosphere. From 1) it follows that some neighbourhood of the point $P_0 \in F^n$ is situated in horoball bounded by horosphere $H^n(P_0)$. Let take dual tangent horosphere $\tilde{H}^n(P_0)$. This horosphere is defined by opposite point at infinity on geodesic line going in the direction of normal n_0 at the point $P_0 \in F^n$, $\tilde{H}^n(\tau)$ are parallel horospheres $\tilde{H}(0) = \tilde{H}^n(P_0)$, $M_\tau = F^n \bigcap \tilde{H}^n(\tau)$, τ is a distance from the horosphere $\tilde{H}^n(P_0)$. Let $f = \tau/_{F^n}$ is the restriction the function τ on the hypersurface F^n. For the function f the point P_0 is a strong local minimum for small τ the set $M_\tau = F^n \bigcap \tilde{H}^n(\tau)$ is a diffeomorphic to the sphere S^{n-1} and bounds on F^n the domain D_τ homeomorphic a ball and contains unique critical point P_0 of the function $f = \tau/_{F^n}$. On the horosphere $\tilde{H}^n(\tau)$ the set M_τ bounds convex domain homeomorphic a ball. Really, the normal ν to M_τ on $\tilde{H}^n(\tau)$ has the form

$$\nu = \lambda_1 n_1 + \lambda_2 N,$$

where n_1 is unit normal to $\tilde{H}^n(\tau)$, N is a normal to $F^n, \langle \nu, n_1 \rangle = 0$. Therefore

$$\nu = \langle n_1, N \rangle n_1 + N.$$

Let X be the unit vector field tangent to M_τ. Then

$$\langle \nu, \nabla_X X \rangle = \langle n_1, N \rangle \mu + k_n/_{F^n},$$

where μ is the normal curvature of the horosphere $\tilde{H}(\tau)$. Since $k_n/_{F^n} \geqslant k_2$ and $\mu \leq k_2$,then $\langle \nu, \nabla_X X \rangle > 0$,that is the second quadratic form M_τ on $\tilde{H}^n(\tau)$ is a positive definite and the domain on $\tilde{H}^n(\tau)$ bounded M_τ is a convex domain homeomorphic a ball.

Let's consider the body $Q(\tau)$, bounded $\mathcal{D}_\tau$ and $\tilde{H}^n(\tau)$ for small τ. At any boundary point there exists a local supporting horosphere. It is a global supporting horosphere too. And the body $Q(\tau)$ is situated in the horoball bounded by supporting horospheres. Other words the body $Q(\tau)$ is h-convex. Let τ^* be a supremum τ, for which the body $Q(\tau)$ is h-convex, $\mathcal{D}^* = \bigcup \mathcal{D}_\tau$. Let's show $\tau^* = \infty$. Let us assume the contrary. There are three possible cases:

a). $\mathcal{D}^* = F^n$;

b). $\mathcal{D}^* \neq F^n$ and on the boundary S^* of the domain $\mathcal{D}^*$ there are critical points the function $f = \tau/F^n$.

c). $\mathcal{D}^* \neq F^n$ and S^* doesn't contain the critical points the function f.

The case c) is impossible. Really for $\tau > \tau^*$ the set M_τ is homeomorphic the sphere too. It bounds the convex domain on $\tilde{H}^n(\tau)$ and at any boundary point $Q(\tau)$ there exists a local supporting horosphere. It follows that $Q(\tau)$ is a h-convex set for $\tau > \tau^*$ and τ^* is not supremum.

At the case b) the set S^* contains a critical point P of function f. At point $P \in S^*$ the horosphere $\tilde{H}^n(\tau^*)$ is the tangent supporting horosphere to $F^n, S^* \subset \tilde{H}^n(\tau^*) \bigcap F^n$ is the boundary of the convex domain homeomorphic a ball on $\tilde{H}^n(\tau^*)$. Let show that $\tilde{H}^n(\tau^*)$ is the tangent horosphere at all points S^*. Really, some neighbourhood U of the point $P \in F^n$ lies at one side with respect to $\tilde{H}^n(\tau^*), U \bigcap S^*$ belongs $\tilde{H}^n(\tau^*)$. If the horosphere $\tilde{H}^n(\tau^*)$ isn't tangent in some point $Q \in U \bigcap S^*$ then U doesn't lies for one side $\tilde{H}^n(\tau^*)$. The set S^* is homeomorphic to the sphere S^{n-1} and the sets of the points of S^*, such that the horosphere $\tilde{H}^n(\tau^*)$ is tangent, is opened and closed at the same time. This set isn't empty and coincides with S^*. Let $Q(\tau^*)$ be the body bounded $\mathcal{D}^*$ and the domain with boundary S^* on $\tilde{H}^n(\tau^*)$. It is a compact h-convex body with smooth boundary. Let $S(r)$ be the circumscribed sphere $Q(\tau^*)$.

Suppose that a tangent point $P \in S(r)$ to the boundary $Q(\tau^*)$ belongs to $\tilde{H}^n(\tau^*)$. At this case the sphere $S(r)$ is supporting to the

horosphere $\tilde{H}^n(\tau^*)$ at the point P. The sphere $S(r)$ and $\tilde{H}^n(\tau^*)$ are tangent at the point P and convex sides have the same direction. This is impossible.

Let $Q_0 \in \mathcal{D}^*$ be a tangent point of the sphere $S(r)$. For Hadamard manifolds are true the following.

Лемма. 1 *Let $S(r)$ and $S(R)\,(r < R)$be tangent spheres at the point Q in Hadamard manifold of the sectional curvatures $K \leq 0$.*

Suppose at the point Q the convex sides of the spheres are the same. Then at the point Q the normal curvatures of the sphere $S(R)$are less than normal curvatures the sphere $S(r)$ in corresponding directions.

Proof. Let take in M^{n+1} the spherical system coordinate with pole O, where O is the centre of the sphere $S(R)$. In the neighbourhood of the point Q the sphere $S(r)$ has the following parametrization

$$t = h(\theta^1, \dots, \theta^n),$$

where $t, \theta^1, \dots, \theta^n$ are spherical coordinates in M^{n+1} with metric

$$ds^2 = dt^2 + g_{ij}(t, \theta) d\theta^i d\theta^j.$$

The normal curvature $S(r)$ at the tangent point Q of the spheres in the direction $b = (b^1, \dots, b^n)$ is equal:

$$k_n = \frac{\left(\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} - \dfrac{1}{2}\dfrac{\partial g_{ij}}{\partial t}\right) b^i b^j}{g_{ij} b^i b^j} =$$

$$k_n(b)/_{S(R)} + \frac{\dfrac{\partial^2 h(\theta^1, \dots, \theta^n)}{\partial\theta^i \partial\theta^j} b^i b^j}{g_{ij} b^i b^j}. \quad (21)$$

Let take a map:

$$\exp_O^{-1} : M^{n+1} \to T_O M^{n+1} = E^{n+1}.$$

The image of the sphere $S(R)$ is the sphere $\bar{S}(R)$ with the center $\bar{\mathrm{O}} = \exp^{-1}(\mathrm{O})$ and radius R. The image of the sphere $S(r)$ lies in a closed ball in E^{n+1} of radius r with the center $\bar{P} = \exp_{\mathrm{O}}^{-1}(P)$, where P is the center of the sphere $S(r)$.

Really, let consider triangles $OPX, \bar{O}\bar{P}\bar{X}$, where $X \in S(r)$, $\bar{X} = \exp_{\mathrm{O}}^{-1}(X)$; $OP = \bar{O}\bar{P} = R - r, \quad OX = \bar{O}\bar{X} = h$ and $\angle POX = \angle \bar{P}\bar{O}\bar{X}$. From nonpositivity of the sectional curvature M^{n+1} and comparison theorem for triangles if follows that $\bar{P}\bar{X} \leq PX$. In spherical system of coordinates with pole $\bar{O}$ the metric E^{n+1} has the form

$$ds^2 = dt^2 + G_{ij}(t,\theta)d\theta^i d\theta^j.$$

The normal curvature of the image of the sphere $S(r)$ at the point $\bar{Q}$ is equal

$$\overline{k}_n = \frac{\left(\dfrac{\partial^2 h(\theta^1,\dots,\theta^n)}{\partial\theta^i\partial\theta^j} - \dfrac{1}{2}\dfrac{\partial G_{ij}}{\partial t}\right) b^i b^j}{G_{ij}b^i b^j} = \frac{\dfrac{\partial^2 h(\theta^1,\dots,\theta^n)}{\partial\theta^i\partial\theta^j} b^i b^j}{G_{ij}b^i b^j} + \frac{1}{R^2}. \quad (22)$$

As the image $S(r)$ lies in a closed ball of radius r with center $\bar{P}$, then

$$\bar{k}_n \geqslant \frac{1}{r}.$$

From (22) it follows at the point Q

$$\frac{\partial^2 h(\theta)}{\partial\theta^i\partial\theta^j} b^i b^j > 0 \quad (23)$$

From (21) and (23) we obtain the statement of the lemma1.

∎

It follows from lemma that normal curvatures of the horosphere less than normal curvatures of the tangent sphere which lies inside horoball, bounded by horosphere. Therefore at the point $Q_0 \in F^n$ normal curvatures F^n satisfy an inequality:

$$k_n/F^n \geqslant k_n/S(r) > k_n/H^n,$$

where H^n is the supporting tangent horosphere. But this contradicts the assumption that at any point F^n there exists the direction a such that

$$k_n(a)/_{F^n} = k_n(a)/_{H^n}.$$

And the case b) is impossible. The case a) is possible only for $\tau^* = \infty$, otherwise it is true arguments of the case b).

We have proved that any tangent horosphere is globally supporting.

Let P_1, P_2 be different arbitrary points F^n and tangent supporting horospheres H_1, H_2 are different too. Then F^n belongs to intersection of horoballs bounded by horosphere H_1, H_2. Intersection of horoballs is a compact bounded set if the sectional curvature of Hadamard manifold

$$K_\sigma \leq -k_1^2 < 0.$$

Therefore $\tau^* < \infty$, but it is impossible. Hence horosphere H_1 and H_2 coincide and F^n is a horosphere in Hadamard manifold M^{n+1}.

3) Let introduce the horospherical system of coordinates with base F^n in the manifold M^{n+1}. The metric of the ambient space has the form (3). For $t = 0$ we obtain the hypersurface F^n. Principal curvatures of horosphere $t = \mathit{const}$ satisfy the inequalities $k_2 \geqslant \lambda_i \geqslant k_1$, such that the sectional curvature of M^{n+1} satisfies inequality

$$-k_1^2 \geqslant K \geqslant -k_2^2.$$

By condition of the theorem the principal curvatures of F^n satisfy inequality $\lambda_i \geqslant k_2$ and we obtain that $\lambda_i = k_2$ and horosphere F^n is an

umbilical hypersurface. Principal curvature of equidistant horospheres $t = const$ satisfy the Riccati equation.

$$\frac{d\lambda}{dt} = \lambda^2 + K_\sigma,$$

where K_σ is the sectional curvature in the direction of twodimensional plane span on the normal to horosphere and corresponding principal direction. Since $K_\sigma \geqslant -k_2^2$, that

$$\frac{d\lambda}{dt} \geqslant \lambda^2 - k_2^2, \qquad \lambda(0) = \lambda_0.$$

Solving this inequality we obtain

$$\lambda \geqslant k_2 \frac{(k_2 + \lambda_0)e^{-2k_2 t} - (k_2 - \lambda_0)}{(k_2 + \lambda_0)e^{-2k_2 t} + (k_2 - \lambda_0)}$$

for $\lambda_0 = k_2, \quad \lambda \geqslant k_2$, from another side $\lambda \leq k_2$. And we get $\lambda = k_2$ for all values t.

Therefore the coefficients of metric tensor g_{ij} of the ambient space M^{n+1} satisfies the equations:

$$-\frac{1}{2}\frac{\partial g_{ij}}{\partial t} = k_2 g_{ij}.$$

And $g_{ij}(\theta, t) = g_{ij}(\theta, 0)e^{-2k_2 t}$. The metric M^{n+1} has the form

$$ds^2 = dt^2 + e^{-2k_2 t} d\sigma^2,$$

where $d\sigma^2$ is the metric of the base horosphere F^n. Let show that metric of F^n is flat. Suppose that in some point of F^n on some twodimensional plane the sectional curvature $\gamma_2 \neq 0$. Then the sectional curvatures of the coordinates horosphere $t = const$ in corresponding point and direction is equal $\gamma_2 e^{2k_2 t}$. From Gauss formula we get that the sectional curvature of the ambient space M^{n+1} at the same direction is equal

$$\gamma_2 e^{2k_2 t} - k_2^2, \quad -\infty \leq t < +\infty.$$

As the sectional curvature M^{n+1} satisfies the inequality

$$-k_1^2 \geqslant K \geqslant -k_2^2,$$

that $\gamma_2 = 0$ and the manifold M^{n+1} is a space of constant curvature $-k_2^2$.

Proof of the theorem 2. From the part I) of theorem 1 it follows that F^n is a compact convex hypersurface diffeomorphic to S^n. Analogical, to the proof of the theorem 3.1 [9] we obtain that every tangent sphere of radius r_0 is globally supporting and F^n belongs to closed balls bounded of this spheres. It is possible two cases:

I). There exist two different points $P_1, P_2 \in F^n$ such that tangent spheres $S_1(r_0), S_2(r_0)$ at these points of radius r_0 don't coincide. Than F^n lies in intersection of balls bounded of these spheres. In Hadamard manifold the intersection of different balls of radius r_0 belongs to the ball of radius less r_0.

II). At all points F^n the tangent sphere of radius r_0 is the same and F^n coincides with the sphere of radius r_0. Analogical to the proof of part II).3) of theorem1 we obtain that the ball bounded of this sphere isometric to a ball of radius r_0 in Lobachevsky space of curvature $-k_2^2$.

References.

[1] *J.Hadamard*, "Sur certaines propriétes des trajectories en dynamique S. Muth. Pures. Apple, 3 (1897), 331-387 p.

[2] *S.S.Chern, R.K. Lashof,* "The total curvature of immersed manifolds Amer.S. Math., (1957), 79, 308-318 p.

[3] *S.Alexander*,"Locally convex hypersurfaces of negatively curved spaces Proc.Amer.Math.Soc., (1977), 64, N2, 321-325 p.

[4] *S.Heijenort,*"On locally convex manofolds Comm.Pure and Applied Mathematics, (1952), 5, 223-242 p.

[5] *R.Currier,*"On hypersurfaces of hyperbolic space infinitesimally supported by horospheres Trans. Amer. Math. Soc., (1989), 313, N1, 412-431 p.

[6] *A.A.Borisenko, D.I. Vlasenko,*"Convex surfaces in Lobachevsky space Math.physics, analysis, geometry, (1997), 4, N3, 278-285 p.

[7] *S.A.Scherbakov,* "Regularity of radial field on Hadamard manifold which clamped sectional curvature and bounded geometry VINITI, dep. N3815-81, (1981), 30 p.

[8] *A.A.Borisenko*, "Convex sets in Hadamard manifolds Differential Geometry and its Applications, (2002), 17, 111-121 p.

[9] *A.A.Borisenko, V.Miquel,* "Comparison theorems on convex hypersurfaces in Hadamard manifolds Ann.Global Anal. Geom., (2002), 21, 191-202.

[10] *L.P.Eisenhart,*"Riemannian geometry Princeton University Press. Princeton, (1949).